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Research Report No. EM - 17

NUMERICAL METHODS FOR EVALUATION OF THE INTEGRALS FOR VIRTUAL HEIGHT

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Abstract

The Gauss-Christoffel formula is proposed for the numerical evaluation of integrals involving the Chapman distribution. The method is applied to calculate the virtual height of reflection of vertical incidence pulses as a function of frequency for both the ordinary and extraordinary rays in the cases of transverse or longitudinal propagation. It is also shown that the Chapman distribution may be approximated very closely by a cosine curve.

It is indicated how the calculated values of the virtual height may be used to analyze experimental data.

ERRATA

Research Report No. EM-17 - Numerical Methods for Evaluation of the Integrals for Virtual Height - B. Friedman

1. Page 3 -Equation 1 - change "x" to "X" in the denominator
2. Page 7 -First equation from top -

   Remove exponent "2" for denominator of third term.
3. Page 7 - Second equation from top -

   Add exponent "2" for denominator of second term.
4. Page 8 - Equation 20

   Add exponent "2" in denominator inside square root sign to read

   $$\left(R^2 - Ch(x)\right)^2 - R_t^2 R^2$$
5. Page 9 - Equation 28

   Change "d" to read "dx" for variable of integration
6. Page 12- In expression for f(x) for L_1

   Remove $\sqrt{}$ from $R(R \pm R_L)$ which occurs as factor in denominator.
7. Page 16- In last equation

   Change "R^2" to read $Ch(x_o)$
8. Page 23- Interchange labeling of axes.
9. Page 26- Interchange headings of second and third columns.

[illegible]

as a function of the frequency, it is [illegible] integrals involving a Chapman ionization distribution which are obtained from the Appleton-Hartree formula. Unfortunately, the required integrals are so complicated that it is impossible to evaluate them analytically. This paper presents a method for numerical evaluation of integrals involving a Chapman distribution.

The method is used to obtain the virtual height for vertical incidence [illegible] First, when there is no magnetic field [illegible] the direction of propagation [illegible]. Both the ordinary and extraordinary rays are considered. The results are compared with those of Jaeger* who used a series expansion to evaluate these integrals only in the case when no magnetic field is present. The agreement between the two different methods is good. Recently Hacke** proposed another method for evaluating the integrals by approximating the Chapman distribution by a double parabola. This procedure permits him to express the integrals in terms of [illegible] functions.

In this paper the [illegible] integrals [illegible] for the virtual heights under different propagation conditions are reduced to a standard form in Sections 2 and 3. The integrals are then transformed in Section 4 to permit rapid computation by the Gauss-Christoffel quadrature formula.

It will also be shown that the Chapman distribution can be approximated by a cosine curve and then the resulting integrals can be evaluated very simply by an adaptation of the Gauss-Christoffel quadrature formula.

In Section 5 a method developed by Booker and Seaton, and applied [illegible] for the determination of the heights of maximum ionization density [illegible] and to study [illegible] the ionization [illegible] a Chapman distribution. [illegible] results agree with those obtained by him using a more [illegible] procedure.

---

*Jaeger, J. C., Proc. Phys. Soc. 59, 87 (1947).

**Hacke, J. E., Jr., I.R.E. Proc. 36, 724 (1948)

The method is also extended to the case [illegible] [illegible] [illegible] ionic drift. An analysis of data obtained from Pennsylvania State College provides an illustration of a useful method of checking the validity of the Chapman hypothesis.

In Section 6 tables and graphs of the virtual height as a function of frequency are given for several cases. The data for the longitudinal case is for a few propagation conditions of local interest. No data have been included for the case of transverse propagation. The detailed computational work in this report is [illegible] and [illegible] [illegible] rather than complete.

[illegible] Formula for Virtual Height

[illegible] also composed of waves of a circular frequency, ω, [illegible], which is propagated [illegible] shall [illegible] integral expression for the virtual height of [illegible] which this wave is reflected.

Let μ be the index of refraction in the ionosphere. According to the Appleton-Hartree theory, if absorption is neglected, μ is given by [illegible][1]

$$\mu^2 = 1 - \frac{1}{[illegible]}$$

where

$X = \omega_0^2/\omega^2$, [illegible]

e, m = charge and mass of electron

N = number of ions in the ionosphere

$Y_T = \omega_T/\omega$, $\omega_T = \mu_0 H_T\, e/m$

μ_0 = magnetic permeability of the atmosphere

$Y_L = \omega_L/\omega$, $\omega_L = \mu_0 H_L\, e/m$

H_T = transverse component of the earth's magnetic field,

H_L = longitudinal component of the earth's magnetic field.

[illegible] the positive sign gives the ordinary ray, while the negative sign gives the extraordinary ray.

---

[1] Appleton, E. V., [illegible]
Hartree, D. R. [illegible]

We assume that N varies with height according to the Chapman distribution[1]:

(2) $$N = N_0 \exp \tfrac{1}{2}\left[1 - z - e^{-z}\right]$$

where N_0 is the maximum ion density of the layer, [illegible] above the earth in scale units defined by

$$z = \frac{h - h_m}{H}$$

h_m = height of maximum ionization

h = actual height above the earth

$H = \frac{kT}{mg}$ = scale height

Let ω_c be the critical frequency, then ω_c [illegible] the wave which will be reflected by the maximum if the magnetic field is neglected. We now define

(3) $$X = \frac{\omega_0^2}{\omega^2} = \frac{\omega_c^2}{\omega^2}\,\frac{[illegible]}{[illegible]} = a\,\phi(z)\ [illegible]$$

where

(4) $$a = \omega_c/\omega \ [illegible]$$

We also introduce the notation

(5) $$y_H = \omega_H/\omega_c, \quad y_L = \omega_L/\omega_c$$

(6) [illegible]

---

[1] Chapman, S. [illegible]

We shall now obtain an integral for the time of travel [illegible] vertically [illegible]. The time, dt, to travel a distance, dz, is given by

$$dt = dz/u$$

where u is the group velocity of the [illegible] in the [illegible] and travelling [illegible] light [illegible] downwards. The total time [illegible] given by

$$\int_0^{z} dz/u,$$

which [illegible] times the velocity of light in vacuo the virtual height is defined as

$$(1) \qquad h' = c\int_0^{h} dz/u = h + \int_0^{h} \left(\frac{c}{u} - 1\right) dz$$

The second term [illegible] later [illegible] integral [illegible] not [illegible] when we introduce the Chapman distribution.

From the well-known formula for the group velocity:

$$u = 1/(dk/d\omega)$$

and the fact that

$$(2) \qquad v = c/\mu \qquad \text{and} \qquad v = \omega/k$$

so that

$$k = \omega\mu/c$$

we find that

$$(3) \qquad c/u = \frac{d(\mu\omega)}{d\omega} = [illegible] = \frac{1}{\mu}\,\frac{d(\mu^2\omega^2)}{d(\omega^2)}$$

Hence

$$h' = \int [illegible] \, dz$$

Consider, now, the case of transverse [illegible] at $y_1 = 0$. Then, for the ordinary ray, equation [illegible] reduces to

$$[illegible] \qquad \mu^2_{ot} = 1 - X = 1 - \frac{Ch(r)}{n^2}$$

[illegible], this formula is the same as in the case when no [illegible].

[illegible]

$$(12) \qquad \mu^2_{[illegible]} = 1 - [illegible]$$

$$= 1 - \frac{Ch(r)}{n^2} - \frac{n_1^2\, Ch(r)}{n^2 (n^2 - n_1^2 - Ch(r))}$$

[illegible] n^2 is the only quantity dependent on ω and Ch(r) is [illegible], [illegible] n_1, as previously defined, is a con[illegible] ratio of the gyro-magnetic frequency [illegible].

In order to find the group velocity [illegible] evaluate

$$(13) \qquad \frac{d}{d(\omega^2)}(\mu^2\omega^2) = \frac{d(\mu^2 n^2)}{d(n^2)}$$

[illegible] for the ordinary ray, from equation (11)

$$\mu^2_{ot}\, n^2 = n^2 - Ch(r)$$

so that [illegible]

$$(14) \qquad [illegible]$$

In equation [illegible] put

$$n^2 = n_c^2 + Hx' , \qquad [illegible] = H\,dx'$$

then

$$h'_t = x_0 + \int^{x_c} \left[\frac{1}{\mu_{ot}} - 1 \right] dx = [illegible]$$

[illegible] the value of h' for [illegible] is

$$[illegible]$$

Here [illegible] height [illegible] corresponds to the ground.

For the [illegible] tion [illegible]

$$\mu^2_{xt} R = R' \cdot Ch(x) - \frac{R_t\, Ch(x)}{\left(R^2 - R_t^2 - Ch(x)\right)^2}$$

and

$$\frac{d}{dx}\left(\mu_{xt}\, r^2\right) = 1 - \frac{R_t^2\, Ch(x)}{\left(R^2 - R_t^2 - Ch(x)\right)}$$

so that

(17) $$x'_{xt} = \int_{-\infty}^{x_0} dx \left\{ \frac{1}{\mu_{xt}} \left[1 - \frac{[illegible]}{\left(R^2 - R_t^2 - Ch(x)\right)} \right] - 1 \right\} [illegible]$$

[illegible] value of x for which

$$\mu_{xt} = 0$$

[illegible]

[illegible]

If we solve this equation, we find

$$Ch(x_0) = F(R^2) [illegible]$$

and since Ch(x) is always positive, it is clear that we must distinguish between the cases where R_t is greater or less than R.

Suppose, first, that $R_t > R$, then from (12)

(18) [illegible]

(19) [illegible] = 2(R + [illegible]) [illegible]

Note that [illegible] will penetrate the layer.

Now, consider [illegible] then from (12) we may [illegible]

[illegible]

and

$$\rho'_{xt} = 0$$

when

(21) $$Ch(x) = 2(R - R_t).$$

If $R_t = R$, then

$$\frac{1}{\rho(x)} = \pm \frac{1}{\sqrt{2R^2 - Ch(x)}}$$

[illegible]

$$\rho_{xt} = 0 \text{ when}$$

$$Ch(x) = R^2$$

Re-write formula (17) as

(22) $$x'_{xt} = T_1 + R_t^2\, T_2 + x_0$$

where

(23) $$T_1 = \int_{-\infty}^{[illegible]} \frac{1}{\rho_{xt}} [illegible]$$

(24) $$T_2 = \int_{-\infty}^{[illegible]} \frac{[illegible]}{\rho_{xt}\,(R^2 - R_t^2 - Ch(x))^{[illegible]}}$$

[illegible] is similar [illegible] In this case F_L [illegible]

$$\mu_L^{2} = 1 - \frac{1}{1 + F_L} [illegible]$$

to give

$$\mu_L^{2} = R^{2} [illegible] \frac{R_L\,Ch(x)}{2(R + R_L)}$$

and

[illegible]

Finally

(25) $$x_L' = \int_{-\infty}^{x_0} dx \left\{ \frac{1}{\mu_L}\left[1 + \frac{R_L\,Ch(x)}{2(R + R_L)^{2} [illegible]}\right] - 1 \right\} + x_0$$

where x_0 is the value of x for which $\mu_L = 0$. We rewrite (25) as

(26) [illegible]

where

(27) $$I_1 = \int_{-\infty}^{0} dx \left[\frac{1}{\mu_L} - 1\right]$$

and $$I_2 = \int_{-\infty}^{x_0} \frac{Ch(x)\,dx}{\mu_L} .$$

Since $$\mu_L = \sqrt{1 - \frac{[illegible]}{(R + R_L)}}$$

is the same as the value of μ_L in (15) if $F^{[illegible]}$ is replaced by [illegible] values obtained in the calculation of (15) can be [illegible] values of (27).

[illegible] Integrals in Standard Form

The integrals required for computing the virtual heights are given in section [illegible] by equations (15), [illegible], (24), (27) and (28). To facilitate the computation these integrals will be transformed to the standard form:

$$I = \int^{x_0} \frac{f(x)\ \mathrm{Ch}(x)\ dx}{\sqrt{1 - \mathrm{Ch}(x)/\mathrm{Ch}(x_0)}}$$

where x_0 [illegible] form for each [illegible] propagation of the ordinary ray is obtained as follows.

From equation (13) and (15)

$$h'_0 = \int_{-\infty}^{x_0}\left[\frac{1}{\sqrt{1 - \mathrm{Ch}(x)/H^2}} - 1\right]dx = \int_{-\infty}^{x_0} \frac{1 - \sqrt{1 - \mathrm{Ch}(x)/H^2}}{\sqrt{1 - \mathrm{Ch}(x)/H^2}}\,dx$$

$$= \int_{-\infty}^{x_0} \frac{\mathrm{Ch}(x)\,dx}{H^2\left(1 + \sqrt{1 - \mathrm{Ch}(x)/H^2}\right)\sqrt{1 - \mathrm{Ch}(x)/H^2}}$$

The integral is in the standard form if we take,

$$f(x) = \frac{1}{H^2\left(1 + \sqrt{1 - \mathrm{Ch}(x)/H^2}\right)} = \frac{1}{\mathrm{Ch}(x_0)\left(1 + \sqrt{1 - \mathrm{Ch}(x)/\mathrm{Ch}(x_0)}\right)}$$

while taking account of equation (16) shows x_0 is given by $\mathrm{Ch}(x_0) = H^2$.

We can then summarise our results as follows:

Transverse Propagation

Ordinary Ray

$$h'_{0t} = x_0 + I_0$$

$$\mu^2_{0t} H^2 = H^2 - \mathrm{Ch}(x), \quad \mathrm{Ch}(x_0) = H^2$$

and I_0 is of the form I with

$$f(x) = \frac{1}{[illegible]}$$

Orthogonal ray

[illegible]

$$\frac{[illegible]}{[illegible]} = \sqrt{\frac{R_1^2 - R^2 - Ch(x)}{R_1^2 [illegible]}}\;\frac{[illegible]}{\sqrt{R^2 - Ch(x)}} = \frac{[illegible]}{[illegible]} \; ; \; Ch(x_0) = R(R + R_1)$$

and I_1 is of the form I with

$$f(x) = \frac{R^2 - Ch(x)}{(\sqrt{A} + \sqrt{B})\sqrt{R_1 R - R^2 + Ch(x)}\;\sqrt{Ch(x_0)}}$$

and the integral I_2 is of the form I with

$$\frac{R\sqrt{R_1^2 + Ch(x) - R^2}}{(R^2 - R_1^2 - Ch(x))\sqrt{R_1 R - R^2 + Ch(x)}\;\sqrt{Ch(x_0)}}$$

[illegible]

$$\frac{1}{\sqrt{\cdot}} = R\sqrt{\frac{R^2 - R_1^2 - Ch(x)}{(R^2 - Ch(x))^2 - R_1^2 R^2}} = R\sqrt{\frac{C}{D}} \; ; \; Ch(x_0) = R(R - R_1)$$

and I_1 is of the form I with

$$f(x) = \frac{R^2 - Ch(x)}{(R\sqrt{C} + \sqrt{D})\sqrt{R_1 R + R^2 - Ch(x)}\;\sqrt{Ch(x_0)}}$$

The integral I_2 is of the form I with

$$f(x) = \frac{R_1\sqrt{R^2 - Ch(x) - R_1^2}}{(R^2 - R_1^2 - Ch(x))\sqrt{R_1 R + R^2 - Ch(x)}\;\sqrt{Ch(x_0)}}$$

Longitudinal Propagation

$$\mu_L^2 = [illegible]$$

$$\mu_L^2 = 1 - \frac{[illegible]}{R(B \pm [illegible])} \; ; \; [illegible](x_0) = R(B \pm R_L)$$

L_1 is of the form I with

$$f(x) = \frac{[illegible]}{\sqrt{R(1 \pm R_L)}\left(1 + \sqrt{4 - [illegible]}\right)}$$

L_2 is of the form I with

$$f(x) = 1$$

The upper sign corresponds to the ordinary ray the lower sign corresponds to the extraordinary ray.

4. Calcul[illegible]

[illegible]

In this section the Gauss-Christoffel quadrature formula will be applied to evaluate the integrals for virtual heights. The virtue of this particular method is that by using only n points it gives an approximation to the [illegible] which is exact for a polynomial of degree 2n-1 and the weight function w(x) [illegible], chosen to take care of the singularity of the integrand. The Gauss-Christoffel theorem[1] is as follows:

Let w(x) be a nonnegative integrable function in the interval (a,b) and $p_n(x)$ ($n = 0, 1, 2, \ldots$) the orthonormal polynomials of degree $0, 1, 2, \ldots$ respectively, which are orthonormal in the interval (a,b) with the weight function w(x), that is

$$(1) \qquad \int_a^b p_n(x)\, p_m(x)\, w(x)\, dx = \delta_{nm} \quad \begin{matrix} = 0 & m \neq n \\ = 1 & m = n \end{matrix}$$

and g(x) be an arbitrary polynomial of degree 2n-1 at most, then the Gauss-Christoffel formula states that

$$(2) \qquad \int_a^b w(x)\, g(x)\, dx = \sum_{i=1}^{n} \lambda_i\, g(x_i)$$

where $x_1, x_2, \ldots, x_n$ are the roots of $p_n(x) = 0$ [illegible] and λ_i are the Christoffel numbers.

We reproduce here the usual proof which [illegible] directly the method for computing the λ_i and the polynomials $p_n(x)$. Since g(x) is a polynomial of degree 2n-1 at most, then by ordinary division

$$(3) \qquad g(x) = q(x)\, p_n(x) + r(x)$$

where q(x) and r(x) are polynomials of degree n-1 at most. By the orthonormality property $p_n(x)$ is orthogonal to any polynomial of degree less than n, so

$$\int_a^b w(x)\, q(x)\, p_n(x)\, dx = 0$$

Also, we may write

$$(4) \qquad r(x) = a_0\, p_0(x) + \ldots + a_{n-1}\, p_{n-1}(x)$$

---

[1] For another proof and further details on this method c.f. Szegö, G., Orthogonal Polynomials, Amer. Math. Soc. Colloq. Pub. Vol. XXIII, 1939, p. [illegible]. [illegible] & v. Mises, R., Differential- und Integralgleichungen der Mechanik und Physik, Vol. I, p. 36[illegible].

polynomial of degree n-1 at most. Now

$$(5)\quad \int_a^b g(x)\,w(x)\,dx = \int_a^b r(x)\,w(x)\,dx = a_0 \int_a^b p_0(x)\,w(x)\,dx\,,$$

and $p_0(x)$ is a known constant which may be determined from $\int_a^b w(x)\,dx$ [illegible]. In order to find a_0, consider the identity obtained from (4) and (2)

$$(6)\qquad g(x) = q(x)\,p_n(x) + \sum_{k=0}^{n-1} a_k p_k(x)$$

Let $x = x_1, \ldots, x_n$ be the n roots of $p_n(x) = 0$, then

$$(7)\qquad g(x_j) = \sum_{k=0}^{n-1} a_k p_k(x_j)$$

which are n linear equations for the n unknowns $a_0, a_1, \ldots, a_{n-1}$. If these are solved for a_0, we find

$$a_0 = \sum_{j=1}^{n} \mu_j\, g(x_j)\,,$$

and then if we set $\lambda_j = \mu_j \int_a^b p_0(x)\,w(x)\,dx$ we have from (5)

$$(8)\quad \int_a^b g(x)\,w(x)\,dx = \sum_{j=1}^{n} \mu_j\, g(x_j) \int_a^b p_0(x)\,w(x)\,dx = \sum_{j=1}^{n} \lambda_j\, g(x_j)\,.$$

To find the coefficients λ_j we may choose particular polynomials $g_j(x)$, for

$$g_j(x) = \frac{p_n(x)}{p_n'(x_j)(x-x_j)}$$

and substituting in (8). It is clear that

$$g_j(x_k) = 0 \quad \text{for } k \neq j \text{ and } g_j(x_j) = 1$$

so that

$$(9)\qquad \lambda_j = \int_a^b \frac{p_n(x)}{p_n'(x_j)(x-x_j)}\,w(x)\,dx.$$

The polynomials $p_n(x)$ may be found by solving a set of linear equations. Let $p_n(x) = \sum_{k=0}^{n} b_k x^k$, then by the orthogonality property

$$\int_a^b x^i\,p_n(x)\,w(x)\,dx = 0 \qquad i = 0, 1, \ldots, n-1$$

and we find that

$$\sum_{k=0}^{n} b_k \int_a^b x^{k+i}\,w(x)\,dx = 0, \qquad i = 0, 1, \ldots, n-1$$

The solution of this set of n equations will determine the ratios of the n+1 coefficients, b_0, b_1, ... b_n. Their absolute values may be determined by using the condition that

$$\int_0^\infty p_n(x)^2 \, w(x) \, dx = 1$$

It will be noted from equation (9) that a constant factor multiplying $p_n(x)$ will not affect the values of the x_i. Neither in the present nor in later applications is it required that the $p_n(x)$ be normed. It is accordingly convenient for finding the values of the coefficients b_k to take $b_n = 1$.

1.2 Application to the Integrals of Virtual Height

Now consider the integral

$$h = \int_0^{x_0} \frac{f(x) \, m(x) \, dx}{\sqrt{1 - m(x_0)/m(x)}}$$

which is the general form of the integrals [illegible]. We shall transform it to a form to which the [illegible] formula is applicable.

Let $y = e^{-x/2}$, then

$$m(x) = e^{1/2} \, y \, e^{-y^2/2} \,, \quad dx = -2 \, dy/y$$

and

$$h = 2 \, e^{1/2} \int_{y_0}^{1} \frac{y \, e^{-y^2/2} \, f(y)}{\sqrt{1 - [illegible]}} \, \frac{dy}{y}$$

$$= 2 \, e^{1/2} \int_{y_0}^{1} [illegible] \left[[illegible] \right]$$

Let $y^2 - y_0^2 = r^2$, then $y \, dy = r \, dr$

$$h = 2 \sqrt{2} \, e^{1/2} \, e^{-y_0^2/2} \int [illegible]$$

According to the [illegible] quadrature formula [illegible] constants q_k, t_k (k=1, ...,n) such that

$$\int_0^\infty e^{-t^2} g(t)\,dt = \sum_{k=1}^{n} q_k\, g(t_k)$$

if g(t) is a polynomial of degree [illegible]. We shall use the right side of this formula [illegible] the integral on the left in case g(x) is not a polynomial.

Consider the set of polynomials $p_1(t)$, $p_2(t)$, ... of degree 1,2,... respectively, which are mutually orthogonal on the interval (0,∞) using the weight function, $w(t) = e^{-t^2}$; then the constants t_k (k = 1,2,...,n) [illegible] $p_n(t) = 0$. The first three are as follows and may be readily [illegible] from the results in the previous section:

$$p_1(t) = [illegible]$$

$$p_2(t) = [illegible]$$

$$p_3(t) = t^3 + 2at^2 + [illegible]$$

where

$$[illegible] = \frac{(\pi-3)(\pi-[illegible])}{3\sqrt{\pi}\,(\pi-[illegible])}$$

The constants q_k are given by the formula (9) of Section [illegible]

$$q_k = \int_0^\infty \left[[illegible] \right] e^{-t^2}\,dt.$$

Using this quadrature formula and remembering that

$$Ch(x_0) = y_0 \exp(1 - y_0^2)/2$$

[illegible] that

$$I = \frac{2\sqrt{2}\,R^2}{[illegible]} \sum_{k=1}^{n} q_k \left[\frac{f(x)\sqrt{y - y_0}}{y\sqrt{1 - \frac{Ch(x)}{Ch(x_0)}}} \right]_{t_k}$$

where the bracket is to be evaluated at the point t_k.

[illegible] polynomials [illegible] the Hermite polynomials the range of the [illegible] while for the Laguerre polynomials the weight function is [illegible]

[illegible]

c_1 [illegible] c_4 [illegible]

c_2 [illegible] c_5 [illegible]

$c_3 = 0.0150$ $c_6 = 1.7917$

[illegible] approximation

[illegible] in Section [illegible] may be evaluated [illegible] approximation to [illegible] distribution. [illegible] be obtained by [illegible]

[illegible] Chapman distribution, we fit a cosine curve to the Chapman [illegible] and get the following:

(1) $$Ch(x) \approx \frac{1}{2}\left(1 + \cos \frac{\pi x}{2x_0}\right) \quad [illegible] \geq x \geq -2x_0 .$$

[illegible] the [illegible] values of [illegible] that [illegible] is [illegible]

[illegible]

$$I = \int_{-\infty}^{[illegible]} \frac{[illegible]}{\sqrt{1 - \frac{Ch(x)}{Ch(x_0)}}}$$

[illegible] is related to the function $f(x)$ in Section 3.2 by the relation [illegible] we use the approximation (1), I becomes

[illegible]

Let

(2) $$1 + \cos \frac{\pi x}{2x_0} = Ch(x_0)\,(1 + [illegible])$$

and substitute in the integral. Then,

$$(4) \qquad J = \frac{2e^{[illegible]}\sqrt{2}}{\pi} \int_0^{\pi} \frac{g(x)\, d\theta}{\sqrt{\frac{2-\mathrm{Ch}(x_0)}{\mathrm{Ch}(x_0)} - \cos\theta}}$$

This integral is [illegible] integral but we shall evaluate [illegible] in Section 4.1.

The weight function w(x) will be taken as unity, but instead of using the ordinary algebraic polynomials we shall use trigonometric polynomials. The trigonometric functions orthogonal over the interval 0 to π are cos nθ (n = 0,1,2,...), which may in fact be expressed as polynomials of degree n in cos θ. The zeros of the n^{th} polynomial are then simply

$$\theta_j = \frac{2j-1}{2n}\pi$$

In this case the orthogonal polynomials are

$$p_k = \sqrt{\frac{2}{\pi}} \cos k\theta$$

$$p_0 = \frac{1}{\pi}$$

Formula (7) of Section 4.1 represents n equations when j = 1,2,...,n. The coefficients λ_j may be obtained by the following special device. If the equations are added together and use is made of the identities

$$\sum_{j=1}^{n} \cos k \frac{2j-1}{2n}\pi = 0 \qquad k = 1,2,\ldots,n$$

we find

$$a_0 = \frac{\pi}{n} \sum_{j=1}^{n} g(x_j)$$

Then by comparison with equation (8) of Section 4.1 the λ_j can be seen to be

$$\lambda_j = \pi/n \qquad j = 1,2,\ldots,n$$

The integral is then

$$(5) \qquad J = \frac{2e\sqrt{2}}{n} \sum_{k=1}^{n} \frac{g(x_k)}{\sqrt{[illegible] - \cos\theta_k}}$$

where [illegible],

$$[illegible] \, (x_k)\,(1 \; [illegible])$$

We may express ([illegible]) completely in terms of $Cn(x_0)$ by using [illegible] for each

$$Cn(x_k) = \frac{Cn(x_0)}{2}\,(1 + \cos \alpha_k)$$

or

(6) [illegible]

In particular [illegible]

$$Cn(x_1) = 0.957\, Cn(x_0)$$

(7) $$Cn(x_2) = 0.500\, Cn(x_0)$$

$$Cn(x_3) = 0.0[illegible]\, Cn(x_0)$$

and (6) [illegible] be expressed [illegible]

[illegible]

[illegible] may be used [illegible] height with frequency. [illegible]

[illegible] of the maximum [illegible]

Consider, first, the [illegible] from the critical frequency, f_c, [illegible]

(1) [illegible]

[illegible]

(2) [illegible]

$$= h_1 + H\phi(R)$$

where ϕ [illegible] function [illegible] given in Fig. 3.

[illegible] (2) [illegible]

[illegible]

[illegible]

and since $\phi(.93)$ [illegible] $\phi(.98)$ [illegible]

$$h'(.93 f_c) - h_1 \; [illegible] \; h_1 - h'(\ldots f_c)$$

[illegible] $\phi(R)$. From Equation (2) [illegible] f_c.

[illegible]

[illegible]

See [illegible] (1940).

[illegible] case of [illegible] propagation. Let [illegible] be the [illegible] corresponding [illegible] respectively, then

(3) [illegible]

where ω_N is determined by [illegible]

(4) [illegible] and ℓ_0 [illegible]

from [illegible]

(5) ω_N^2 [illegible]

[illegible] (4) N_m may be determined. From Equation (26) of Section 2,

$$h' = [illegible]\ \Psi_1(R,R_L)$$

$$h'_2 = h_m + H\ \Psi_2(R,R_L)$$

where

$$\Psi_1(R,R_L) = [illegible]$$

$$\Psi_2(R,R_L) = L_1 + \frac{1}{2(R+R_L)}L_2 + x_{0L}$$

[illegible] plotting h' against $\Psi_1(R,R_L)$ will give h_m and H, and hence [illegible] may also be obtained by plotting h' against $\Psi_2(R,R_L)$. A comparison of the values obtained for h_m and H should give an indication of [illegible] Chapman distribution [illegible] in the layer.

We shall illustrate the method by analyzing some data given by [illegible]. [illegible] the data [illegible] the following table.

R = f/f$_c$	1.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
h' in km	250	2[illegible]	[illegible]	260	254	265	300	317	345

If h' is plotted against φ(R) as in the graph, we observe that all the points except the first two lie on a straight line. From the graph we find

[illegible]

[illegible]

The above value of [illegible] agrees [illegible] with the value of [illegible], which was found by Henning. Also, the fact that the points lie on a straight line indicates that the assumption of a Chapman distribution of the ionization is justified. It should be noted that if the observed ionization is not distributed according to the Chapman theory, then the above analysis is not valid.

When these methods of interpretation are applied to the analysis of [illegible] showing a magneto-ionic split, a series of tests for the validity of the Chapman distribution is obtained. Suppose the data for the ordinary ray are plotted against [illegible]. The first test is whether the resulting curve is a straight line. If it is, the values of [illegible] are read off. Now, [illegible] extraordinary ray [illegible] plotted against $\psi_2(E,E_m)$. The second test is that this curve be a straight line also. If it is, the values of h_m and H on this curve are read off. The third test is that the values of H and h_m for the two curves should be the same.

These tests were carried out on some experimental data obtained at the Ionosphere Propagation Laboratory of Pennsylvania State College under quiet night time conditions. It was assumed that the propagation was nearly longitudinal and that no retardation was caused by the E-layer. The assumption of longitudinal propagation in this location would introduce only a slight error. The results of this analysis* showed that the assumptions of the Chapman distribution for the F-region at that time were justified in this case.

Because this method seems to give a simple way to analyse experimental data on the ionosphere, we shall present the analysis of the Pennsylvania State data in detail. The observed h' – f records are shown in Fig. 6.

The first thing was to find the asymptotes of the curve. This was done by considering the ratio Δh/Δh' and extrapolating it to zero. We find f_1 = 7.38 mc., f_2 = 6.54 mc. Then, solving the equations (3) written in terms of frequency,

$$f_1^2 = f_m^2 + f_m f_H$$

$$f_2^2 = f_m^2 - f_m f_H$$

we find

$$f_m = 6.97; \quad f_H = 0.842$$

---

* Rydbeck, O.E.H., A Theoretical Survey of the Possibilities of Determining the Distribution of the Free Electrons in the Upper Atmosphere, 1942, Trans. Chalmers Univ. of Tech. No. 3.

[illegible].

$$N_1 = [illegible]/0.97 = 0.322 \text{ and } R = [illegible]/6.[illegible]$$

The functions $N_1'(T,\epsilon_1)$ and $N_1'(T,\epsilon_1)$ are either computed from the formulas or [illegible] read from tables [illegible] graphs. There are two possible ways of comparing the experimental data with the theory. One is to plot the values of $N_1'(T,\epsilon_1)$ [illegible] above the corresponding values of R. The experimental curve which gives [illegible] in terms of I can be used to read N' in terms of R. The values of N' are then plotted against the corresponding values of R and the result should be a straight line. The result is shown in Figs. C and D for both the ordinary and the extraordinary ray.

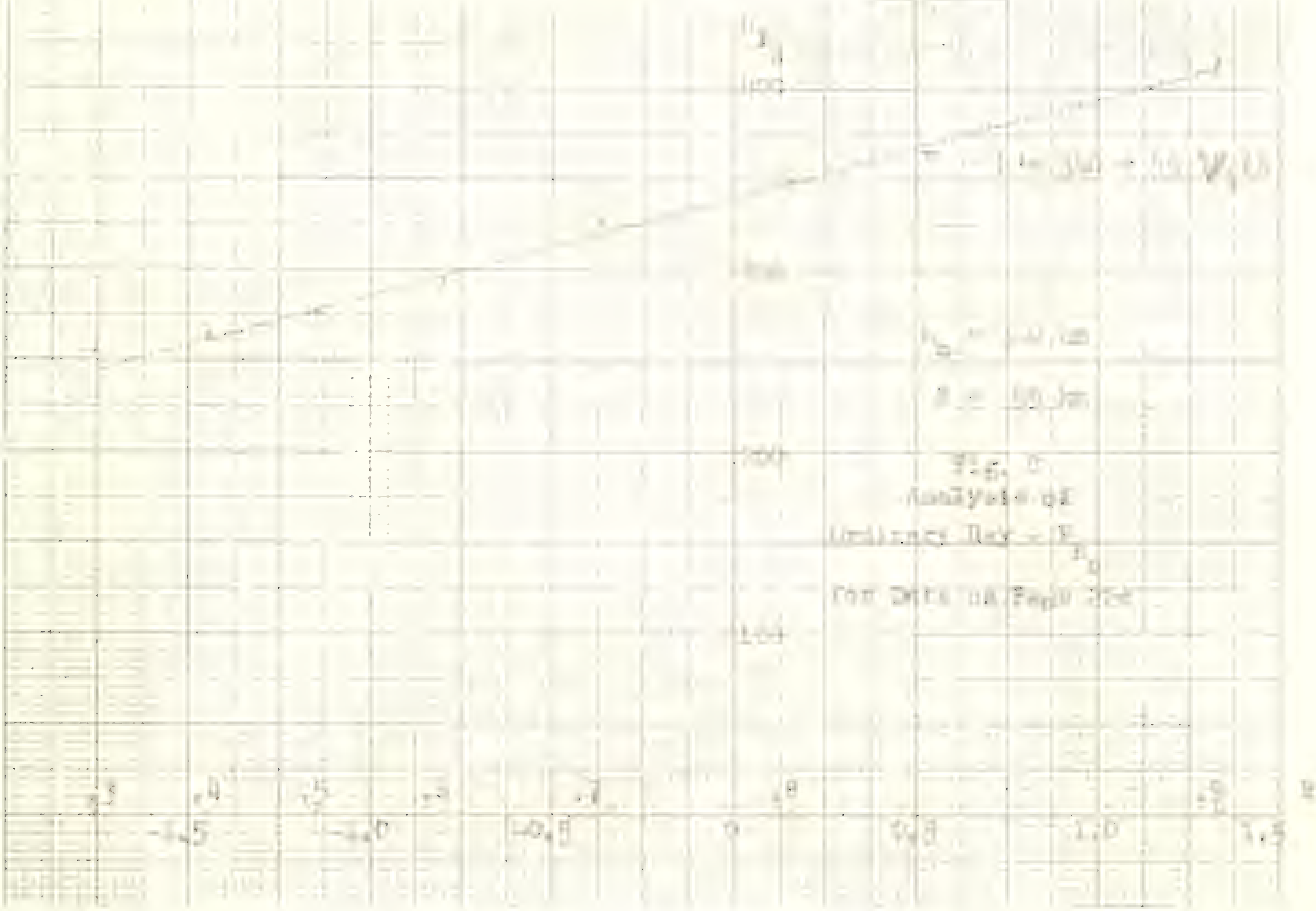


Fig. C
Analysis of
Ordinary Ray - T_{P_O}
for Data on Page [illegible]

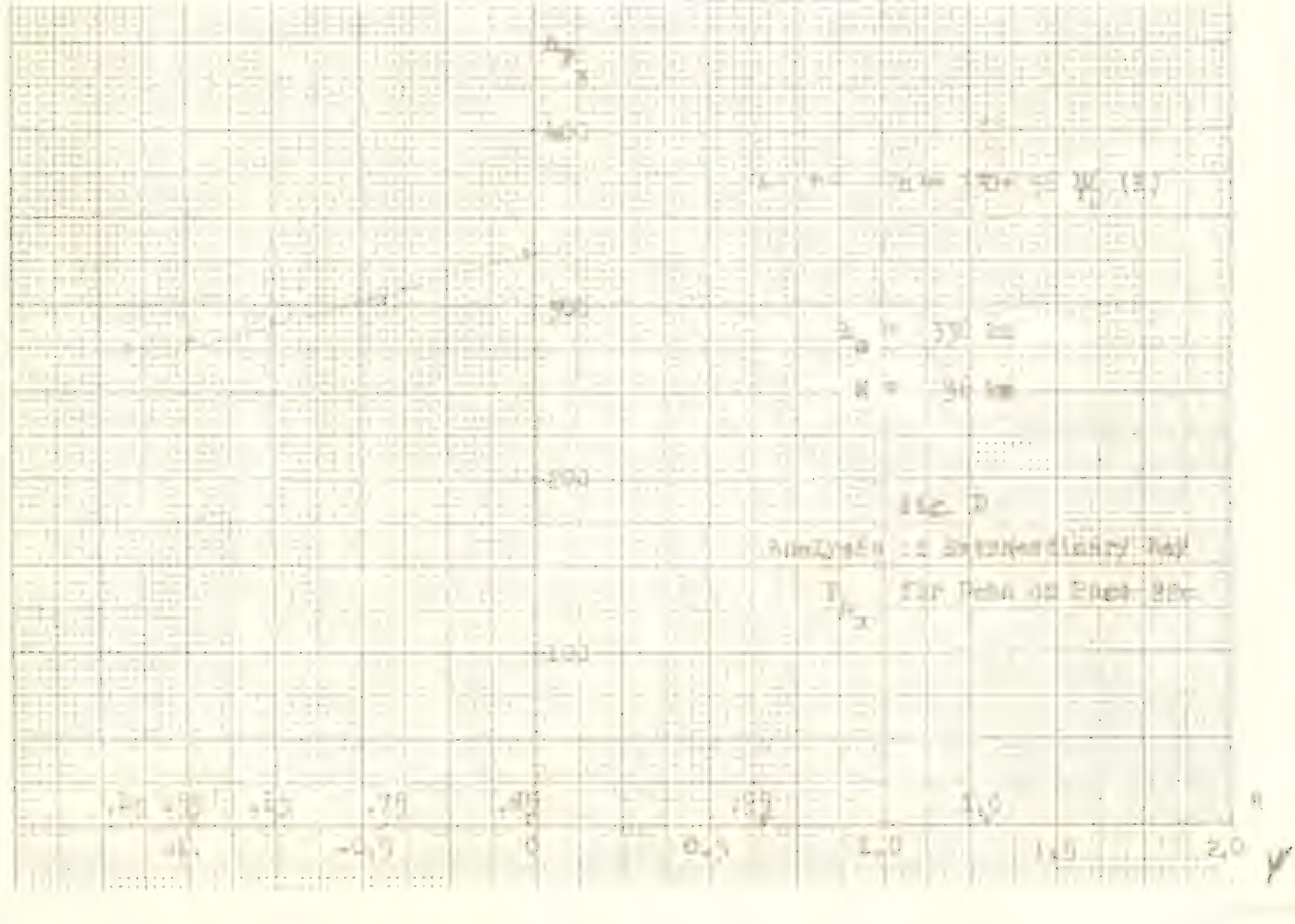


Fig. D
Analysis of Extraordinary Ray
T_{P_X} for Data on Page [illegible]

[illegible] [illegible] [illegible] [illegible] against F in the 1st quadrant and $\psi(R, r_L)$ [illegible] [illegible] [illegible].

[illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] point Q_ψ vertically [illegible] [illegible] the ψ [illegible] [illegible] [illegible] line to the vertical axis and [illegible] to the point Q'_ψ on the negative horizontal axis. The point Q_s with vertical coordinate equal to that of Q_L' and horizontal coordinate equal to that of Q_ψ is a point on the desired [illegible] [illegible]. The values of r_L and [illegible] may be read off.

The following page (Fig. 7) indicates this procedure for the F_L used in Fig. 5 [illegible] [illegible] in the left [illegible] [illegible] with those shown in Fig. [illegible].

TABLE 3

Cosine Approximation to the Chapman Distribution

z	$Ch(z) = \exp(1 - z - e^{-z})/2$	$\frac{1 + \cos \frac{\pi z}{2.6}}{2}$
0	1.0	1.000
- .1	.9970	.9964
- .2	.9891	.9858
- .3	.9753	.9682
- .4	.9550	.9440
- .5	.9287	.9134
- .6	.8944	.8770
[illegible]	[illegible]	[illegible]
- .8	.8062	.7888
- .9	.7553	.7378
-1.0	.6984	.6837
-1.1	.6363	.6270
-1.2	.5712	.5685
-1.3	.5041	.5090
-1.4	.4369	.4493
-1.5	.3712	.3904
-1.6	.3083	.3330
-1.7	.2498	.2781
-1.8	.1969	.2263
-1.9	.1506	.1784
-2.0	.1114	.1350
-2.1	.0795	.0969
-2.2	.0543	.0645
-2.3	.0355	.0383
[illegible]	[illegible]	[illegible]
-2.5	.0130	.0060
-2.6	.0072	.0003
-2.7	.0037	.0017
[illegible]	[illegible]	[illegible]

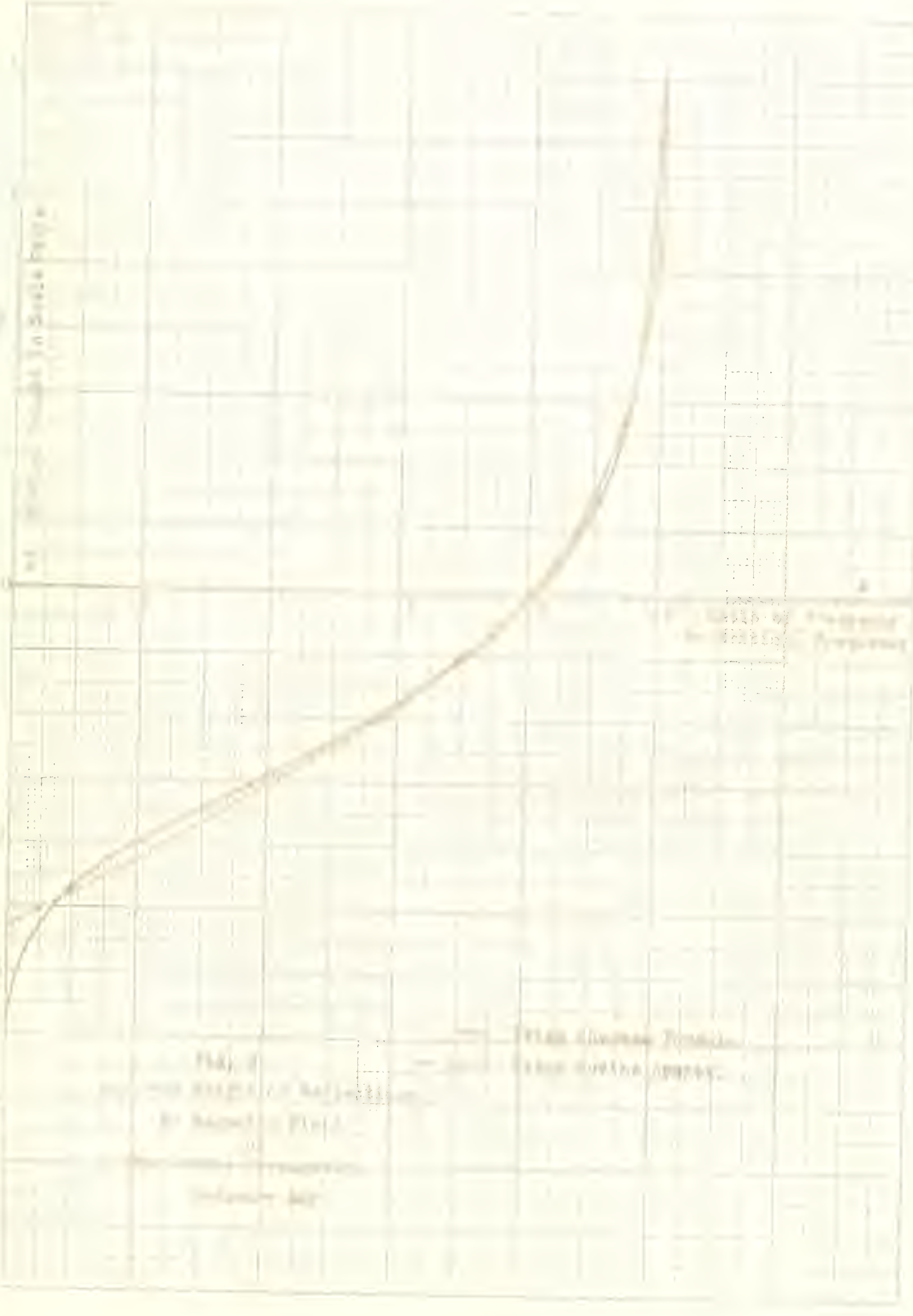


Fig. 8 [illegible] Magnetic Deflection
By Magnetic Field
[illegible]

[illegible] corrections

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] dx

[illegible]

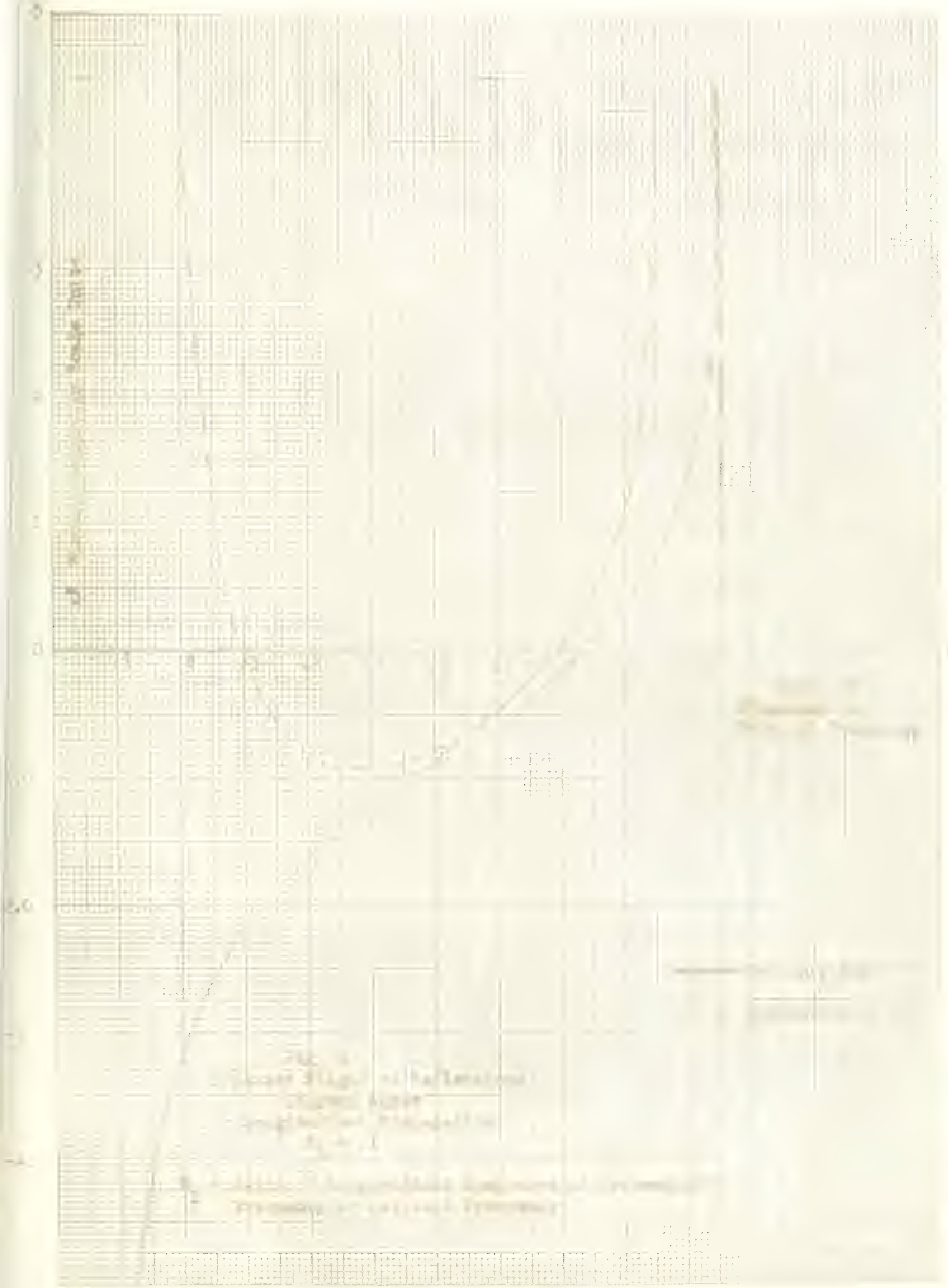
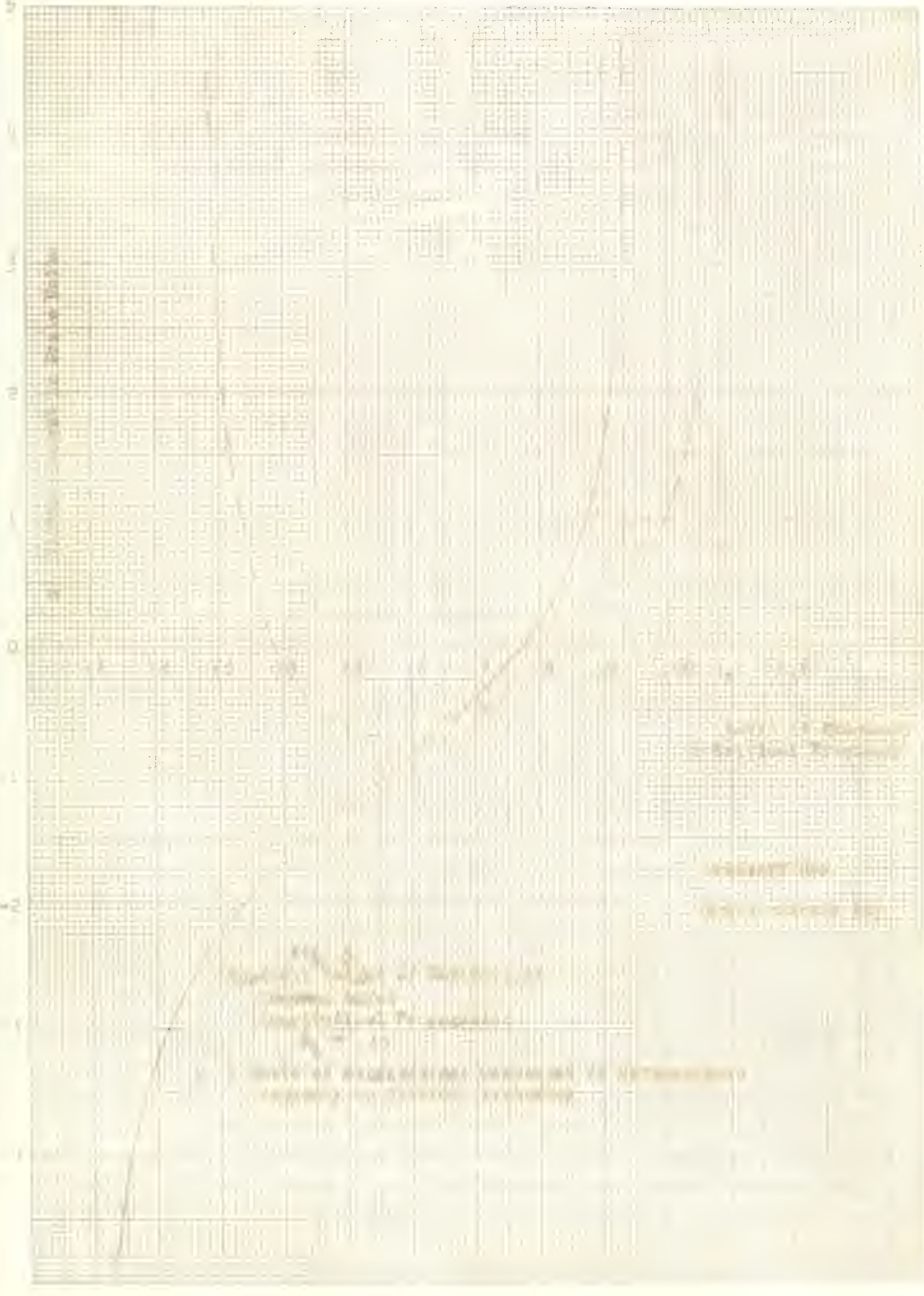
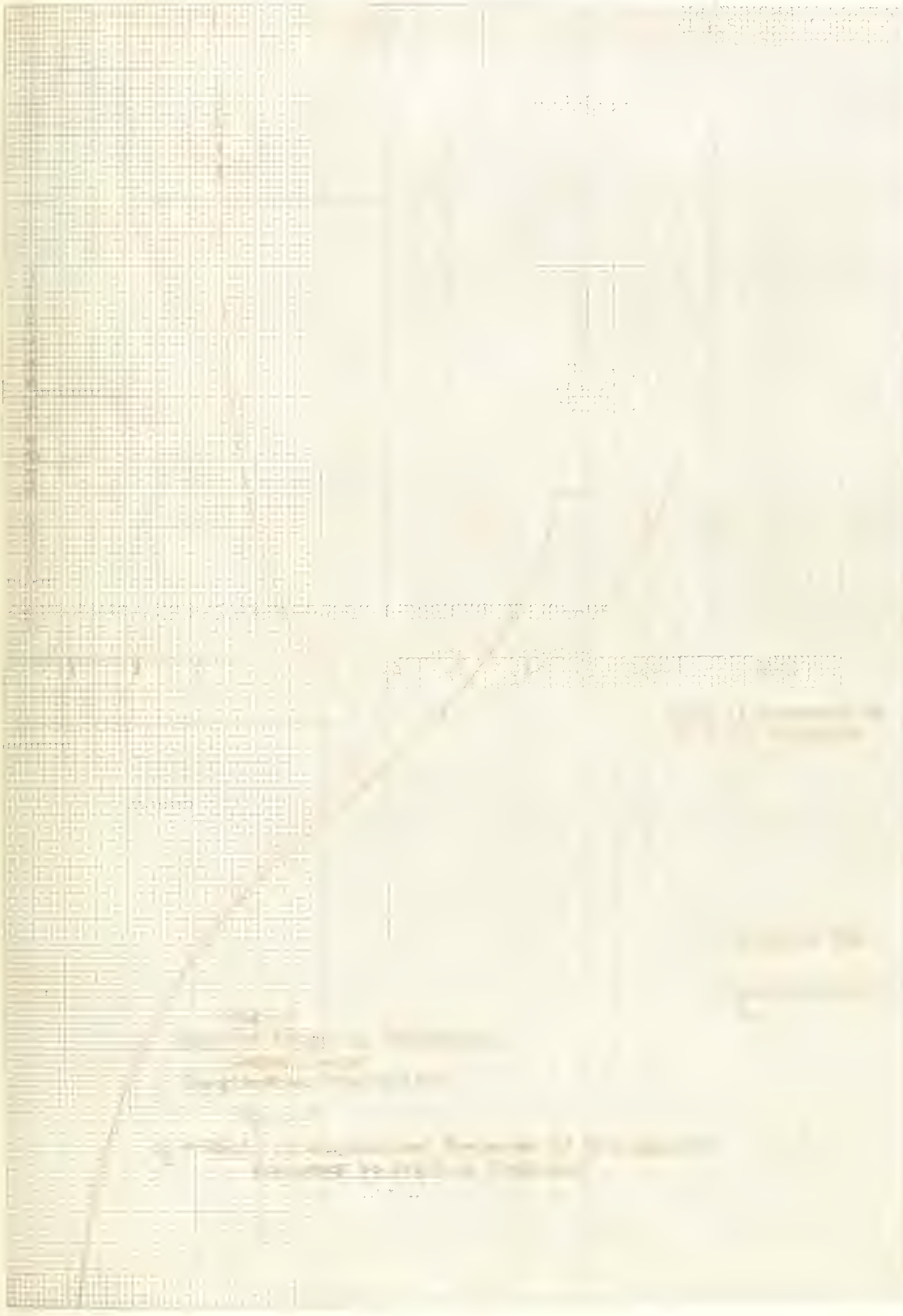


Fig. [illegible]
[illegible]
[illegible]
[illegible]
[illegible]

[illegible]
[illegible]


5
3
2
0
-1
-2



[illegible]

[illegible]

Longitudinal [illegible]

Compressed Layer

Ordinary [illegible]

Table No. 3

[illegible]

$L_1 = \int_{-\infty}^{x_0} dx$ [illegible]

$L_2 =$ [illegible]

[illegible]

	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
y	Fig. 3	Fig. 4	Fig. 5	Fig. 6	Fig. 7	Fig. 8
.1	-6.[illegible]	-6.85	-6.30	-6.21	-6.50	-5.01
.2	-3.16	-3.11	[illegible]	-3.26	-3.20	-3.6[illegible]
.3	-2.17	-2.10	-2.2[illegible]	-2.2[illegible]	-2.27	-2.10
.4	-1.66	-1.6[illegible]	-1.6[illegible]	-1.7[illegible]	[illegible]	-1.5[illegible]
.5	-1.23	-1.25	-1.26	-1.1[illegible]	-1.16	-1.06
.6	-.85	-.87	-.85	-.84	-.7[illegible]	-.80
.7	-.46	-.49	-.41	-.38	-.23	-.52
.72						-.3[illegible]
.74						.92
.76						1.20
.78						2.22
.8	-.0[illegible]	-.13	-.14	-.53	-.74	
.82					1.16	
.84					1.31	
.85					[illegible]	
.865					2.13	
.9	1.14	1.32	2.17	2.17		
.92	1.61	1.91	2.92			
.94	2.13	2.76				
.96	2.35					
.975	3.50					

Apparent Height of Reflections

Longitudinal Propagation

Chapman Layer

Ordinary Ray

Table No. 4

$$x_L' = x_0 + L_1 + \frac{R_L}{2(R - R_L)^{2R}} \; L_2$$

$$L_1 = \int_{-\infty}^{x_0} dx \left[\frac{1}{\mu_L} - 1\right] \qquad\qquad L_2 = \int_{-\infty}^{x_0} \frac{Ch(x)}{L}$$

$$\mu_L^2 = 1 - \frac{Ch(x)}{R(R - R_L)}$$

	x'					
	$R_L = .1$	$R_L = .122$	$R_L = .15$	$R_L = .2$	$R_L = .3$	$R_L = .5$
R	Fig. 3	Fig. 4	Fig. 5	Fig. 6	Fig. 7	Fig. 8
.18	11.65					
.2	6.33					
.25	1.48	3.57	9.30	45.7		
.35	- .58	- .12	.71	3.56	54.4	
.45	- .91	- .77	- .50	.29	5.13	
.55	- .87	- .80	- .67	- .40	.99	65.17
.65	- .66	- .65	- .59	- .48	.04	7.22
.75	- .39	- .37	- .36	- .33	- .12	2.01
.85	+ .08	+ .01	- .01	- .05	+ .03	.82
.95	+ .74	+ .67	+ .56	+ .45	.29	.53
1.00	1.41	1.29	1.04	.84	.57	.58
1.05	4.44	3.04	2.07	1.46	.87	.72
1.15					3.69	1.13
1.25						2.79

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